

Lecture Notes, January 21, 2010, and following

Convexity

A set of points S in R^N is said to be convex if the line segment between any two points of the set is completely included in the set.

S is convex if $x, y \in S$, implies $\{z \mid z = \alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\} \subseteq S$.

S is said to be strictly convex if $x, y \in S, x \neq y, 0 < \alpha < 1$, implies $\alpha x + (1 - \alpha)y \in \text{interior } S$.

The notion of convexity is that a set is convex if it is connected, has no holes on the inside and no indentations on the boundary. A set is strictly convex if it is convex and has a continuous strict curvature (no flat segments) on the boundary.

Economically, this notion corresponds to "diminishing marginal utility" "diminishing marginal rate of substitution" "diminishing marginal product".

Properties of Convex Sets

Let C_1, C_2 be convex subsets of R^N . Then:

$C_1 \cap C_2$ is convex,

$C_1 + C_2$ is convex,

\bar{C}_1 is convex

The Market, Commodities and Prices

N commodities

$x = (x_1, x_2, x_3, \dots, x_N) \in R^N$, a commodity bundle

The market takes place at a single instant, prior to the rest of economic activity.

commodity = good or service completely specified
description
location
date (of delivery)

Price system : $p = (p_1, p_2, \dots, p_N) \neq 0$. $p_i \geq 0$ for all $i = 1, \dots, N$.

Value of a bundle $x \in R^N$ at prices p is $p \cdot x$.

Firms and Production Technology

$F, j \in F, j = 1, \dots, \#F.$

Production technology: $\mathcal{Y}^j \subset \mathbb{R}^N.$

$$y \in \mathcal{Y}^j$$

Negative co-ordinates of y are inputs; positive co-ordinates are outputs.

$$y \in \mathcal{Y}^j, y = (-2, -3, 0, 0, 1)$$

Production function

$$f^j(x) \equiv \max \{ w \mid (-x, w) \in \mathcal{Y}^j \}.$$

The Form of Production Technology

$$\text{P.II.} \quad 0 \in \mathcal{Y}^j.$$

Literally, P.II means that it is possible for the firm to be inactive --- of course you never really see active market behavior from an inactive firm. It ensures that firms are never forced to operate at a permanent loss. Part of the interpretation is that some potential production plans will be dormant until market prices make it attractive to start production, when we'll see potential firms (operating at 0 level of activity) spring into action (producing $y \neq 0$).

$$\text{P.III.} \quad \mathcal{Y}^j \text{ is closed.}$$

$$\text{P.VI} \quad \mathcal{Y}^j \text{ is a bounded set for each } j \in F.$$

P.III and P.VI $\Rightarrow \mathcal{Y}^j$ is compact

Strictly Convex Production Technology

$$\text{P.V.} \quad \text{For each } j \in F, \mathcal{Y}^j \text{ is strictly convex.}$$

Convexity of production technology rules out scale economies. Strict convexity rules out flat segments on the production frontier assuring us of point-valued profit-maximizing behavior.

$p \in \mathbb{R}_+^N, p = (p_1, p_2, \dots, p_N), p \neq 0.$ The profit at prices p of a production plan y is $p \cdot y$ (=value of outputs minus the value of inputs). We define the supply behavior of the firm as

$$\tilde{S}^j(p) = \{y^{*j} \mid y^{*j} \in \mathcal{Y}^j, p \cdot y^{*j} \geq p \cdot y \text{ for all } y \in \mathcal{Y}^j\}.$$

The \sim atop the S here is intended to remind us of P.VI, the boundedness of the set where the profit maximization is taking place.

Theorem 11.1: Assume P.II, P.III, P.V, and P.VI. Let $p \in R_+^N, p \neq 0$. Then $\tilde{S}^j(p)$ is a well defined continuous point-valued function.

Proof:

Well defined: $\tilde{S}^j(p)$ = maximizer of a continuous real-valued function on a compact set.

Point-valued: Strict convexity of \mathcal{Y}^j , P.V. Point valued-ness implies that $\tilde{S}^j(p)$ is a function.

Continuity: Let $p^v \in R_+^N; v = 1, 2, \dots; p^v \neq 0, p^v \rightarrow p^o \neq 0$. Show $\tilde{S}^j(p^v) \rightarrow \tilde{S}^j(p^o)$.

Proof by contradiction. Suppose not. Then there is a cluster point of the sequence $\tilde{S}^j(p^v), y^*$ so that $y^* \neq \tilde{S}^j(p^o)$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$. That is there is a subsequence p^v so that $\tilde{S}^j(p^v) \rightarrow y^*$. We have $p^v \cdot \tilde{S}^j(p^v) \rightarrow p^o \cdot y^*$ and $p^o \cdot \tilde{S}^j(p^o) > p^o \cdot y^*$. But the dot product is a continuous function of its arguments, so for v large, $p^v \cdot \tilde{S}^j(p^o) > p^v \cdot \tilde{S}^j(p^v)$, a contradiction. Hence $\tilde{S}^j(p^v) \rightarrow \tilde{S}^j(p^o)$.
Q.E.D.

Lemma 11.1: (homogeneity of degree 0) Assume P.II, P.III, P.V, and P.VI. Let $\lambda > 0, p \in R_+^N$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$.

The aggregate supply function for the economy as a whole is the sum of the individual supply functions for the firms. That is

$$\tilde{S}(p) \equiv \sum_{j \in F} \tilde{S}^j(p) .$$

4.4 Attainable Production Plans

Definition: A sum of sets \mathcal{Y}^j in R^N , is defined as

$$\mathcal{Y} = \sum_j \mathcal{Y}^j \text{ is the set } \{y \mid y = \sum_j y^j \text{ for some } y^j \in \mathcal{Y}^j\} .$$

Then the Aggregate technology set is defined as $\mathcal{Y} \equiv \sum_{j \in F} \mathcal{Y}^j$. A typical element of \mathcal{Y} is the sum of production plans for each firm, representing a conceivable aggregate production plan for the economy as a whole. Note that $\mathcal{Y}^j \subset \mathcal{Y}$.

The available resources of the economy possibly to be used as initial inputs to production are denoted $r \in R_+^N$. Then the possible achievable production plans of the economy are limited by the available technology \mathcal{Y} and by the available initial resources r .

Definition: Let $y \in \mathcal{Y}$. Then y is said to be attainable if $y + r \geq 0$.

$y \in \mathcal{Y}$ is attainable if $(y + r) \in [\mathcal{Y} + \{r\}] \cap \mathbf{R}_+^N$.

That is, a production plan y is attainable if it is consistent with the economy's technology and if the resources required as inputs to the plan are available initially as the economy's resource endowment.